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A Parallel Between Two Classes of Pricing Problems in Transportation and Economics

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Abstract. In this work, we establish a parallel between two classes of pricing problems that have attracted the attention of researchers in economics, theoretical computer science and operations research, each community addressing issues from its own vantage point. More precisely, we contract the problems of pricing a network or a product line, in order to achieve maximum revenue, given that customers maximize their individual utility. Throughout the paper, we focus on problems that can be formulated as mixed integer programs.

Keywords. Pricing, networks, mixed-integer programming.

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1 Introduction

In this paper, we contrast two families of pricing models that have lived in parallel for some time. The former one is a semi-classical product pricing problem in economics, where the design and pricing policy of a firm must take into account the purchasing behavior of utility-maximizing customers. More recently, Labbé et al. [27] set the problem of devising revenue-maximizing tolls on a multi-commodity transportation network within the framework of bilevel programming, a branch of optimization that deals with mathematical programs whose constraint set is described by an auxiliary problem, and is closely related to Stackelberg games in economics. The aim of this presentation is to provide an overview of results, either theoretical (worst-case complexity), methodological (applications) or numerical (exact or heuristic algorithms), associated with both the original models and variants thereof. Throughout, we highlight the relationships between these models, as well as their treatment by the communities of researchers in operations research, economics, and even theoretical computer science. Focusing on models that can be formulated as mixed-integer programs, we provide their general definition, their framework of application, a summary of their main properties, and their interrelationships.

2 Designing and pricing a set of products

2.1 Three paradigms

This first family of problems that we consider is concerned with the design and pricing of a set of products in a market, for which three paradigms have emerged in the literature: buyer welfare, seller welfare and share-of-choices. All three models fit the following framework: given a set $\mathcal{K}$ of purchasers and a set $\mathcal{I}$ of products, the purchasers' preferences for the various products are described by a utility matrix whose elements are $u_i^k : k \in \mathcal{K}, i \in \mathcal{I}$. Each purchaser selects the product with the largest utility, so far as this utility is positive; otherwise he refrains from buying.

The **Buyer Welfare Problem** consists of determining which subset of products $\mathcal{S} \subseteq \mathcal{I}$ should be introduced into the market so as to maximize the sum of the purchasers' utilities. Given a budget Y, and upon the introduction of binary variables y_i that specify whether a product i is introduced into the market, and x_i^k that specify whether product i is selected by purchaser k, the problem can be formulated

as the mixed integer program:

$$\text{(BWP)} \quad \max_x \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}} u_i^k x_i^k$$

subject to:

$$\sum_{j\in\mathcal{I}} u_j^k x_j^k \geq u_i^k y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{1}$$

$$\sum_{i\in\mathcal{I}} x_i^k \leq 1 \qquad \forall k \in \mathcal{K} \tag{2}$$

$$x_i^k \leq y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{3}$$

$$\sum_{i\in\mathcal{I}} y_i \leq Y \tag{4}$$

$$x_i^k, y_i \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}. \tag{5}$$

Constraints (1) ensure that each purchaser selects a product that maximizes itw own utility, constraints (2) force each purchaser to select at most one product, constraints (3) impose that the products selected are available, while constraint (4) provides an upper bound on the number of products that can be introduced into the market.

Now consider additional parameters v_i^k representing the income perceived by a seller if purchaser $k \in \mathcal{K}$ buys product $i \in \mathcal{I}$. The **Seller Welfare Problem** consists of determining which subset of products $\mathcal{S} \subseteq \mathcal{I}$ should be introduced into the market so as to maximize the seller's revenue, again under the assumption that purchasers maximized their utilities, so far as the latter is positive. This yields the mathematical program

$$\text{(SWP)} \quad \max_x \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}} v_i^k x_i^k$$

subject to constraints (1) to (5).

In the **Share-of-Choices Problem** one considers a set $\mathcal{A}$ of attributes associated with the various products, and a set $\mathcal{J}_a$ of levels associated with each attribute a. A product profile is defined as the assignment of a level to each attribute for each product, and is represented by the vector $p = (j_1, j_2, ..., j_{|\mathcal{A}|})$ of its attribute levels. Further, each purchaser k assigns a value w_{aj}^k to each level j of each attribute a, which are normalized to lie between -1 and 1. Purchaser k selects the product whose profile $p = (j_1, j_2, ..., j_{|\mathcal{A}|})$ has the largest utility $w^k(p) = w_{1j_1}^k + w_{2j_2}^k + ... + w_{|\mathcal{A}|j_{|\mathcal{A}|}}^k$, provided that the latter is positive. The Share-of-Choices Problem consists of determining a product profile p so as to maximize the number of satisfied

purchasers. While customer behavior could be cast within the framework of probabilistic discrete choice theory, we will not consider models that involve explicitly a probabilistic structure. The interested reader could refer to Krieger and Green [26], Shioda et al. [35] or Maddah and Bish [29] for further details on this topic.

2.2 Literature review

For the Buyer Welfare Problem, Green and Krieger [16] restrict the products considered to lie in a subset $\mathcal{S} \subseteq \mathcal{I}$, so that constraint (4) is expressed as $\sum_{i \in \mathcal{I}} y_i = |\mathcal{S}|$. In view of the intractability of enumerating all feasible solutions, the authors propose heuristics based on a greedy approach, as well as Lagrangian relaxation.

Referring to a theoretical study by Cornuéjols et al. [9] for an equivalent Plant Location Problem, the ratio of the greedy over the optimal income (the 'performance ratio') is, in the worst case, $Z_G/Z_O = 1 - \left((|\mathcal{S}| - 1)|\mathcal{S}|^{-1}\right)^{|\mathcal{S}|}$. Hence, as $|\mathcal{S}| \to \infty$, the performance ratio is approximately 63%, and will be higher for smaller values of $|\mathcal{S}|$. Green and Krieger ran simulations on small problems, randomly generated with $|\mathcal{K}| = 100$, $|\mathcal{I}| = 10$ and $|\mathcal{S}| = 4$ or 5. In all cases, the greedy heuristic provided solutions within 8% of optimality, and the optimal solution was obtained in over 50% of the instances.

Whereas these methods are effective for the Buyer Welfare Problem, they do not perform well for the Seller Welfare Problem. According to the authors, neither Lagrangian relaxation nor an exact method can address with some success instances of realistic sizes. Moreover, the greedy heuristic approach can yield very poor results. In the worst case, the performance ratio is $|\mathcal{S}|^{-1}$, and becomes arbitrarily bad when $|\mathcal{S}| \to \infty$. Better results are obtained when the parameters v_i^k are not too different from each other, for all $k \in \mathcal{K}, i \in \mathcal{I}$. Tests on randomly generated instances involving 100 purchasers, 10 products and $|\mathcal{S}| = 4$, show that the seller's greedy heuristic is within 5% of optimality, and provides the optimal solution in 78% of the simulations.

Kohli and Krishnamurti [22] propose a dynamic programming heuristic to solve the Share-of-Choices problem involving a single product. In order to highlight the efficiency of this new approach, tests were run on randomly generated instances involving 100 to 400 purchasers, 4 to 8 attributes, and 2 to 5 levels per attribute. The results are obtained very quickly, and the solution values were always within 9%, and on average within 2%, of the optimum, with an optimal solution identified in 46% cases. The authors also compare their approach to an alternative Lagrangian relaxation heuristic. They conclude that the dynamic programming heuristic outperforms the Lagrangian approach in terms of both computational time and solution values, the Lagrangian results being sometimes as far as 42% from optimality. Their dynamic programming heuristic is also significantly faster than an enumeration procedure.

In [23], Kohli and Krishnamurti prove the $\mathcal{NP}$-hardness of the single-product Share-of-Choices Problem. Based upon a graph representation of the problem, they develop two heuristics based, respectively, on dynamic programming and shortest path computations. Whereas both heuristics have arbitrarily bad worst-case bounds, the dynamic programming solution achieved on average within 2% of optimality (at worst within 12%), while the shortest path solution achieved on average within 6% of optimality (at worst within 13%).

Kohli and Sukumar [24] present dynamic programming heuristics for the Buyer Welfare, Seller Welfare and Share-of-Choices problems, assuming multi-product sets for the latter problem, and a multi-attribute structure similar to that of the Share-of-Choices problem for Buyer and Seller welfare (levels have to be determined for each attribute of each product). The heuristics have been tested on randomly generated instances involving 50 to 150 purchasers, 2 to 4 products, 4 to 6 attributes and 2 to 4 levels per attribute. Empirical results are near-optimal, with performance ratios within 2%, 5% and 2% of optimality , and worst ratios within 4%, 15% and 8%. Optimal solutions were found in 10%, 12% and 30% of cases, respectively.

For multi-attribute Buyer Welfare, Seller Welfare and Share-of-Choices problems, Nair et al. [32] implemented a beam search heuristic, i.e., breadth first search with no backtracking, and breadth limited to a given number of promising nodes in the enumeration tree. On random instances generated as in [24], performance ratios fell within 1% of optimality for all the three problems, and optimal solutions were found in 38%, 58% and 66% cases, respectively. Computing times were roughly half those presented in [24].

Keeping with meta-heuristics, Alexouda and Paparrizos [1] consider genetic algorithms for solving the multi-attribute Seller Welfare Problem, and tested it on random instances involving 100 to 150 purchasers, 2 to 3 products, 3 to 7 attributes and 3 to 6 levels per attribute. The method outperformed the beam search heuristic of Nair et al., both in terms of quality (solution improved by 8% on average, optimal solutions found in 74% of cases) and computing time (three times faster).

Finally, observing that constraints (1) are only active when there exists $j \in \mathcal{I}$ such that $x_j^k = 1$ and $u_j^k < u_i^k$ (then one must have $y_i = 0$), McBryde and Zufryden [31] propose to replace constraints (1) in the Seller Welfare problem by the equivalent:

$$y_i + x_j^k \leq 1 \qquad \forall k \in \mathcal{K}, \forall i, j \in \mathcal{I} : i \neq j, u_i^k > u_j^k.$$

Using a generic mathematical solver, they solved randomly generated instances involving 50 to 100 purchasers and 16 products ($Y = 10$) to optimality very quickly. They also obtained good results for a

particular case in which the seller's income $c_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ does not depend on the products selected by the purchasers, which actually corresponds to a set covering problem. Randomly generated instances involving 100 to 300 purchasers and 64 to 512 products ($Y = 10$) could then be solved to proven optimality in less than three seconds.

2.3 Profit and Bundle Pricing Problems

In a seminal paper, Dobson and Kalish [14] consider an extension of the Seller Welfare Problem, where price variables $\pi_i : i \in \mathcal{I}$ are endogenous, and where the introduction of a product i into the market induces a fixed cost f_i for the seller. The population of purchasers is partitioned into segments, each segment being characterized by its total demand $\eta^k : k \in \mathcal{K}$ and reservation prices $r_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ which provide a measure of the value of a given product to the customer of a given segment. The utility u_i^k attached to a segment $k \in \mathcal{K}$ and product $i \in \mathcal{I}$ is defined as the difference between the reservation price r_i^k and the product price π_i, simply. The **Profit Problem** consists of determining a subset of products $\mathcal{S} \subseteq \mathcal{I}$ and the corresponding product prices leading to a maximum profit for the seller. Note that, in contrast with the model of Green and Krieger [16], the subset of products $\mathcal{S}$ is endogenous. In order to manage the case in which a segment would not buy any product (i.e., if all utilities for a given segment are negative), a 'null' product is introduced. This yields the mixed integer program:

$$\text{(PP)} \quad \max_{p,x,y} \sum_{k\in\mathcal{K}, i\in\mathcal{I}} \eta^k \pi_i x_i^k - \sum_{i\in\mathcal{I}} f_i y_i$$

subject to:

$$\sum_{j\in\mathcal{I}} (r_j^k - \pi_j) x_j^k \geq (r_i^k - \pi_i) y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{6}$$

$$\sum_{i\in\mathcal{I}} x_i^k = 1 \qquad \forall k \in \mathcal{K} \tag{7}$$

$$x_i^k \leq y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{8}$$

$$\sum_{i\in\mathcal{I}} y_i \leq Y \tag{9}$$

$$\pi_0 = 0 \tag{10}$$

$$x_i^k, y_i \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}. \tag{11}$$

To solve this problem, the authors propose a 'reverse greedy heuristic' that exploits the underlying structure of the problem. If variables $x_i^k, y_i : k \in \mathcal{K}, i \in \mathcal{I}$ are fixed (i.e., the subset of offered products and the flows are known), then optimal prices can be found in polynomial time by solving shortest path

problems. The procedure is initialized by setting $x_i^k, y_i : k \in \mathcal{K}, i \in \mathcal{I}$ to the values that maximize the utility of each segment. At each iteration, a segment is reassigned to another product or removed from the market, and prices are updated. The procedure stops when no further improvement can be achieved. The selection criterion for choosing the segment to reassign at each iteration is the seller's profit, i.e., among all segments that prevent the seller from increasing its prices, one selects the one which would lead to the largest improvement of the objective function. The authors evaluate the heuristic performances on small randomly generated instances involving 5 purchaser segments and 4 products, and obtain profit ratios, i.e., ratios of (heuristic profit - worst profit) to (best profit - worst profit), within 10% of optimality.

In a related paper, Dobson and Kalish [15] consider the Buyer Welfare and Profit Problems, and extend their previous work to these variants. First, they formally show that the Buyer Welfare and the Plant Location Problems are equivalent, hence the former is $\mathcal{NP}$-hard. Next, the authors propose heuristics for this problem, including greedy (starts with an empty subset $\mathcal{S}$ of products and adds products one at a time in $\mathcal{S}$), greedy interchange (greedy assignment of purchasers to products, followed by pairwise product interchanges until no improvement is possible), reverse greedy (see above, [14]) and reverse greedy interchange. These are tested on randomly generated problems involving from 20 to 800 purchaser segments and 10 to 80 products. All heuristics perform well, with average ratio of heuristic to Lagrangian upper bound within 10% of optimality, and within 1% in most cases. The greedy and greedy interchange approaches perform better than the reverse greedy interchange method, with optimal solutions found for almost all problems, and worst case ratios always within 1% of optimality. Computing times are negligible.

Next, Dobson and Kalish show that the Profit Problem is $\mathcal{NP}$-hard through a reduction involving the Vertex Cover Problem. For this problem, they consider the reverse greedy heuristic introduced in [14]. They also devise a greedy heuristic that inserts in $\mathcal{S}$, at each iteration, the product that yields the largest improvement of the objective, and where products are considered in a decreasing order of their utilities. Each time a product is introduced in $\mathcal{S}$, corresponding prices are computed for all products so as to maximize the objective function. The procedure halts when no further improvement can be achieved. Evaluated on the same instances as those of Buyer Welfare Problem, the second greedy heuristic performs better on, with ratio of heuristic to Lagrangian upper bound within 8% to 22% of optimality. On large instances, however, it runs slower than the greedy heuristic.

Shioda et al. [36] consider the **Full Profit Problem** in which all products are available ($Y = \infty$, $\mathcal{S} = \mathcal{I}$ and $y_i = 1$ for all i in I), and without fixed costs for the introduction of a product into the market. The authors adapt the algorithms of Dobson et Kalish [14, 15]. They also derive a linear mixed integer model for the problem, for which they produce valid inequalities. The methods seem quite effective, even

if the authors do not provide any quantitative conclusion concerning their preliminary results.

Hanson and Martin [19] consider the Profit Problem (with no fixed costs) for 'global elements', such as data-processing software, possessing several components. The $2^n - 1$ products are then identified with the various component subsets of the global element, where n denotes the number of components. For this **Bundle Pricing Problem**, the notation is similar to that of Dobson and Kalish [14]. Further, they assume that product prices are sub-additive, i.e., whenever product $i \in \mathcal{I}$ is the union of other products, then its price is lower than the sum of the prices of these other products:

$$\pi_i \leq \sum_{j \in S} \pi_j \qquad \forall i \in \mathcal{I}, \forall \mathcal{S} \subseteq \mathcal{I} : i = \cup_{j \in S} j \tag{12}$$

Based on a mixed integer formulation of the problem, problems involving 5 to 10 purchaser segments and 4 components (thus 15 products) have small integrality gaps (2 to 4%) and are solved to optimality very quickly. Whenever the number of components n is large, the authors propose a formulation involving but a limited number of subsets. Indeed they observe that, in this case, there often exists a 'key component' that belongs to all subsets offered in the market, and to which additional components could be appended. Based on a more complex mixed integer formulation adapted to this feature, instances involving up to 4 purchaser segments and 20 components (in addition to the key component) are solved very quickly. This behavior might be a consequence of the limited number of purchaser segments considered in the numerical experiments.

In Guruswani et al. [18], all products of the Bundle Pricing Problem are offered, and sub-additivity is not assumed. The problem is shown to be $\mathcal{APX}$-hard through a reduction to Vertex Cover and a logarithmic approximation algorithm is proposed. Approximation or polynomial time algorithms, together with other theoretical results, are also provided for specific cases. Unfortunately, no numerical results are provided.

Finally, Nichols and Venkataramanan [33] consider a formulation of the **Conjoint Buyer Welfare and Profit Problem** similar to the one suggested by Dobson and Kalish [14, 15], but that involves a weighted objective function including both seller profit and purchaser utility. Three heuristics are proposed for its solution. The first one is a 'pure' genetic algorithm used for comparison purposes. Next, a genetic procedure is used to generate product prices, while utility-maximizing flows $x_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ are obtained by applying a partial enumeration (branch-and-bound) procedure. A third heuristic randomly selects the products to be introduced into the market. Next, for a given set of flows, corresponding prices are obtained by solving an inverse problem. The three heuristics, applied to instances involving 20 to 1000 purchaser segments and 10 to 100 products, show that the modified genetic methods outperform

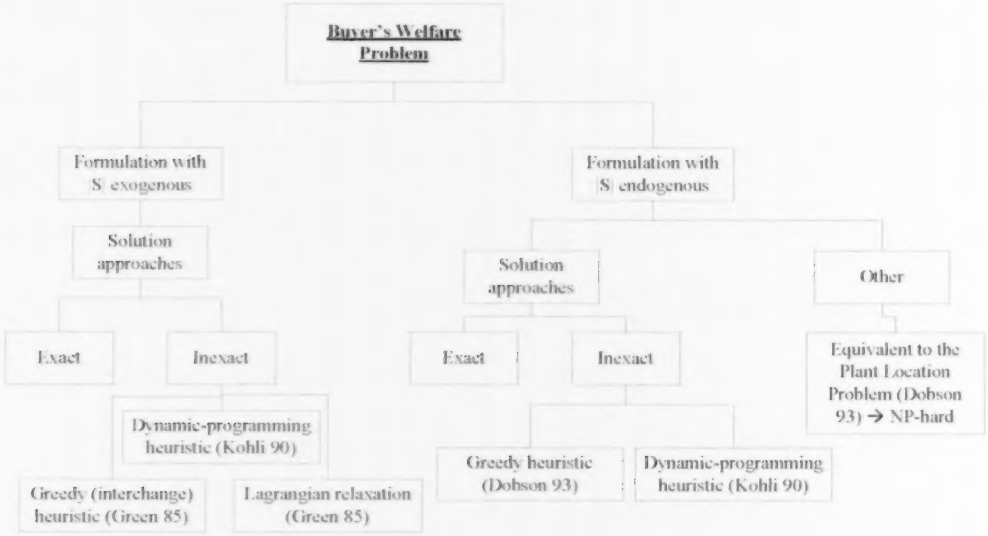

Figure 1: Main contributions to the Buyer's Welfare Problem

the 'pure' one. The results also show that the relaxation methods perform better, on large instances, than a pure genetic algorithm, thus prompting the development approaches where only a subset of 'hard' variables is genetically treated. The authors do not provide further details concerning the performance ratios or the computing times of their algorithms.

We close this section with three figures (Figures 2, 3 and 4) that illustrate a taxonomy of research pertaining to the Buyer Welfare, Seller Welfare and Share-of-Choices Problems, respectively.

3 Network Pricing

In this section, we establish a parallel between the design and pricing problems presented in Section 2 and a network pricing family of problems.

3.1 Highway Network Pricing Problems

In a transportation context, we assume that each purchaser segment $k \in \mathcal{K}$, also called commodity in the sequel, represents a set of users travelling from one node o^k to another d^k in the network while minimizing their costs. Then the set of products $a \in \mathcal{A}$ is the set of all possible paths that could be chosen by commodities. In addition to a fixed cost c_a^k that depends on the commodity k and on the path

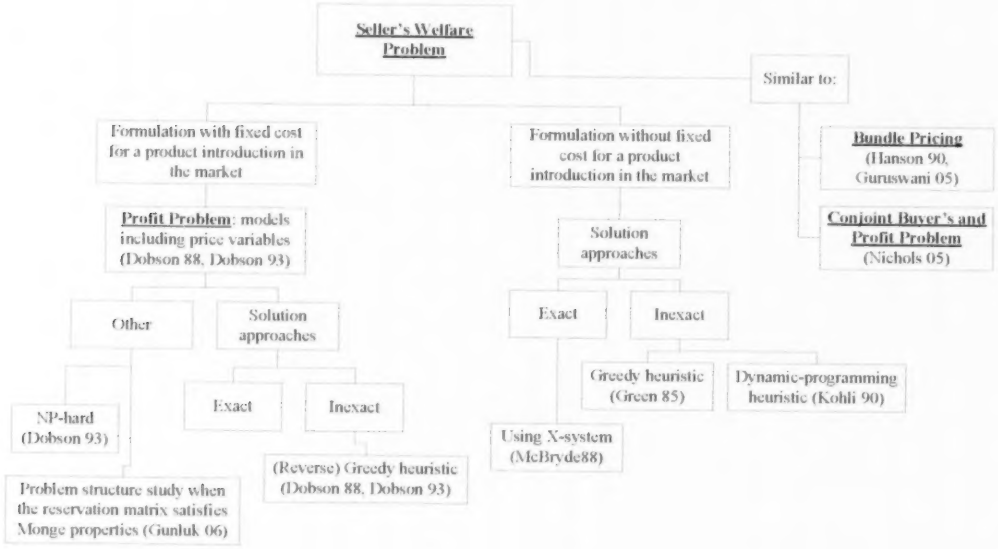
Seller's Welfare Problem
Similar to:
Formulation with fixed cost for a product introduction in the market
Formulation without fixed cost for a product introduction in the market
Bundle Pricing (Hanson 90, Guruswani 05)
Profit Problem: models including price variables (Dobson 88, Dobson 93)
Solution approaches
Conjoint Buyer's and Profit Problem (Nichols 05)
Other
Solution approaches
Exact
Inexact
Greedy heuristic (Green 85)
Dynamic-programming heuristic (Kohli 90)
NP-hard (Dobson 93)
Exact
Inexact
Using X-system (McBryde88)
(Reverse) Greedy heuristic (Dobson 88, Dobson 93)
Problem structure study when the reservation matrix satisfies Monge properties (Gunluk 06)

Figure 2: Main contributions to the Seller's Welfare Problem

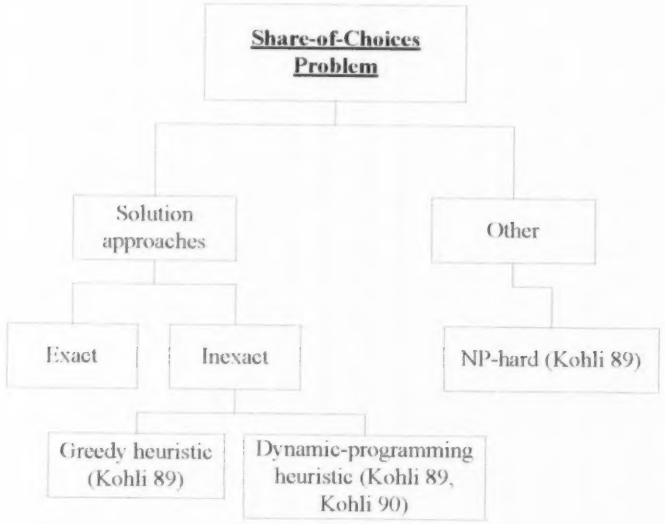

Figure 3: Main contributions to the Share-of-Choices Problem

a, each path a is subject to a toll t_a imposed by an authority which seeks to maximize its revenue. For each commodity k, we assume that an alternative toll free path also exists. Being only subject to a fixed cost c^k_{od}, this path allows to set an upper bound $c^k_{od} - c^k_a$ on the toll level that commodity k would be willing to pay for travelling on the toll path a rather than on the toll free path. These upper bounds correspond to the reservation prices r^k_a in the design and pricing family of problems presented in Section 2.

This fits the framework of a toll highway where tolls are determined with respect to given entry and exit points. We assume that commodities who have left the highway are not allowed to reenter, which implies that paths are uniquely determined by their respective entry and exit nodes. The **Highway Network Pricing Problem** consists of devising the toll levels that should be imposed on the paths of the network so as to maximize the authority's revenue. Reacting to the tolls, each commodity travels on the shortest path from its origin to its destination, with respect to a cost equal to the sum of toll and initial cost.

The work of Dewez [11] and Heilporn [20] focuses on topologies that reflect the features of an actual toll highway. More specifically, they consider structured networks composed by a toll path (the highway) and toll free arcs linking the origin and destination nodes together, as well as to and from the highway. Commodities either travel on the shortest toll free path from their origin to their destination, or take the highway, using shortest toll free paths to and from it. As toll levels are frequently determined with respect to given entry and exit points on the highway, the authors consider a complete toll subgraph where every single feasible path from any origin to any destination in the network contains exactly one toll arc.

Some notation is in order. Consider a multi-commodity network defined by a node set $\mathcal{N}$, an arc set $\mathcal{A} \cup \mathcal{B}$ and a set of origin-destination pairs $\{(o^k, d^k) : k \in \mathcal{K}\}$ for the commodities, each one endowed with a demand η^k. For each toll arc $a \in \mathcal{A}$, let $t(a), h(a) \in \mathcal{N}$ denote its tail and head nodes respectively. For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, let c^k_a denote the fixed cost on the corresponding path $o^k \to t(a) \to h(a) \to d^k$, where $t(a), h(a) \in \mathcal{N}$ are the entry and exit nodes on the highway, respectively. The fixed cost on the toll free path $o^k \to d^k$ is denoted by c^k_{od}, while the corresponding flow variable is x^k_{od}. For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, variable x^k_a represents the flow on the corresponding path $o^k \to t(a) \to h(a) \to d^k$. Further, variable t_a denotes the toll on path a (i.e., toll arc a), while variable p^k_a represents the actual revenue corresponding to commodity k and path a. This

yields the mixed integer linear model (Dewez [11], Heilporn [20]):

$$\text{(HP)} \quad \max_{p} \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k$$

subject to:

$$\sum_{a \in \mathcal{A}} x_a^k \leq 1 \qquad \forall k \in \mathcal{K} \tag{13}$$

$$\sum_{a \in \mathcal{A}} (p_a^k + c_a^k x_a^k) + c_{od}^k (1 - \sum_{a \in \mathcal{A}} x_a^k) \leq c_b^k + t_b \qquad \forall k \in \mathcal{K}, \forall b \in \mathcal{A} \tag{14}$$

$$p_a^k \leq M_a^k x_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \tag{15}$$

$$t_a - p_a^k \leq N_a (1 - x_a^k) \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \tag{16}$$

$$p_a^k \leq t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \tag{17}$$

$$p_a^k \geq 0 \qquad \forall a \in \mathcal{A} \tag{18}$$

$$x_a^k \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \tag{19}$$

where $M_a^k : k \in \mathcal{K}, a \in \mathcal{A}$ and $N_a : a \in \mathcal{A}$ are large constants. By the flow constraints (13), each commodity $k \in \mathcal{K}$ chooses at most one toll path a. By constraints (14), the cost of the optimal path for a commodity $k \in \mathcal{K}$ is smaller than the cost of any other path for this commodity. Next, constraints (15) to (17) ensure that the actual revenue is consistent with the commodity revenue, i.e., $p_a^k = t_a x_a^k$ for all $k \in \mathcal{K}, a \in \mathcal{A}$.

Further, for the sake of realism, triangle and monotonicity constraints (20), (21) can be appended to the problem. The former ensure that it cannot be profitable to leave and re-enter the highway, while the latter forbid the toll along a path to be less than the toll of any subpath, i.e.,

$$t_a \leq t_b + t_c \qquad \forall a, b, c \in \mathcal{A} : \; t(a) = t(b),\ h(b) = t(c),\ h(c) = h(a) \tag{20}$$

$$t_a \geq t_b \qquad \forall a, b \in \mathcal{A} : \; t(a) = t(b) < h(a) = h(b) + 1 \text{ or } t(a) = t(b) - 1 < h(a) = h(b)$$
$$\text{or } t(a) = t(b) > h(a) = h(b) - 1 \text{ or } t(a) = t(b) + 1 > h(a) = h(b) \tag{21}$$

For single commodity problems and in the absence of Monotonicity constraints, the above problem can be solved trivially. Indeed, the toll arc yielding the largest potential revenue for the leader can be found in $O(n)$-time, and the optimal solution consists in setting its toll to the maximum value compatible with the

toll-free path. Other tolls are set to arbitrarily large values. For multi-commodity problems, Dewez [11] proposes a solution approach based on the enumeration of the lower level solutions. Unfortunately, the time required to solve the problem to optimality grows exponentially with the number of commodities and the number of nodes in the network. Alternatively, the author proposes several flow selection heuristics. Once flows are determined, optimal tolls can be recovered through the solution of an inverse problem, which consists of determining revenue maximizing tolls compatible with a given flow assignment. We briefly describe the three most interesting heuristics:

1) For each commodity $k \in \mathcal{K}$, select the toll arc with the largest upper bound $M_a^k : a \in \mathcal{A}$. Next, solve the inverse optimization problem.

2) For each commodity $k \in \mathcal{K}$, select the toll arc with the largest upper bound $M_a^k : a \in \mathcal{A}$. Observe that, if two commodities use the same toll arc, the leader could take advantage to force the use of another toll arc for one of both commodities (with respect to the demand η^k and the upper bounds M_a^k). Next, solve the inverse optimization problem.

3) For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, set $x_a^k = 0$ if the upper bound on the revenue M_a^k is "small" with respect to the upper bound for other commodities. For instance, $M_a^k < \alpha \max_{k' \neq k} M_a^{k'}$ with $\alpha \in (0, 1)$. Next, solve the restricted problem.

When tested on random grids instances, the best heuristics produced solutions within 5% of optimality very quickly.

More recently, Heilporn [20] proves the $\mathcal{NP}$-hardness of the Highway Network Pricing Problem, and derives strong valid inequalities. Next, focusing on two-commodity problems, she show that classes of valid inequalities, as well as classes of constraints in the original formulations, define facets of the convex hull of feasible solutions. In the absence of triangle and monotonicity constraints, a complete description of the convex hull of feasible solutions was obtained for single-commodity problems.

Grigoriev et al. [17] consider a network pricing situation where commodities select at most one toll arc. Since the resulting topology is that of bridges crossing a river, they refer to it as the **Cross River Network Pricing Problem**. The authors prove that this particular problem is $\mathcal{NP}$-hard. They show that uniform pricing yields an n-approximation scheme, where n is the number of toll arcs. Under further assumptions, uniform pricing yields an $O(\log n)$-approximation algorithm.

3.2 A generic Network Pricing Problem

In this section, we address the issue of pricing the arcs of a general transportation network. Specifically, let $\mathcal{A}$ be a subset of toll arcs and $\mathcal{B}$ the complementary subset of toll free arcs. Assuming that, for a

given toll policy $t = (t_a)_{a \in \mathcal{A}}$, network users travel on shortest paths with respect to the sum of tolls and fixed costs $c = (c_a)_{a \in \mathcal{A}}$, the **Network Pricing Problem** consists of devising a revenue maximizing toll policy.

Since tolls are set before flows are assigned, the problem belongs to a class of hierarchical, sequential and non cooperative bilevel optimization programs, where a leader (the authority) integrates within its optimization process the reaction of a follower (the users) to its decisions. Bilevel programming and the related mathematical programs with equilibrium constraints have been the topic of several studies. For recent reviews, we refer to Dempe [10], Marcotte and Savard [30], Colson et al. [8] and Luo et al. [28].

Upon the introduction of vectors $x^k = (x_a^k)_{k \in \mathcal{K}, a \in \mathcal{A}}$ that specify the flows on commodities $k \in \mathcal{K}$ (i.e., $x_a^k = 1$ if commodity k travels on toll arc a, and $x_a^k = 0$ otherwise), the Network Pricing Problem can be formulated as the bilevel program (Labbé et al. [27]):

$$\text{(NP)} \quad \max_{t,x} \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k t_a x_a^k$$

subject to:

$$t_a \geq 0 \qquad \forall a \in \mathcal{A} \tag{22}$$

$$x \in \arg\min_{x} \sum_{k \in \mathcal{K}} \left(\sum_{a \in \mathcal{A}} (c_a + t_a) x_a^k + \sum_{a \in \mathcal{B}} c_a x_a^k \right) \tag{23}$$

subject to:

$$\sum_{a \in i^- \cap \mathcal{A}} x_a^k + \sum_{a \in i^- \cap \mathcal{B}} x_a^k - \sum_{a \in i^+ \cap \mathcal{A}} x_a^k - \sum_{a \in i^+ \cap \mathcal{B}} x_a^k = \begin{cases} -1 & \text{if } i = o^k \\ 1 & \text{if } i = d^k \\ 0 & \text{otherwise} \end{cases}$$

$$\forall k \in \mathcal{K}, \forall i \in \mathcal{N} \tag{24}$$

$$x_a^k \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \tag{25}$$

where i^- (resp. i^+) denotes the set of arcs having node i as their head (resp. tail).

In view of the unimodularity of the constraint matrix associated with the shortest path problem, one may drop the integrality requirements for the flow variables x. The lower level problem can then be replaced by its primal-dual optimality conditions, yielding a single-level program involving complementarity constraints. Through the introduction of actual revenue variables p_a^k, Labbé et al. [27] derive the

mixed integer linear formulation

$$\text{(NP2)} \quad \max \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k$$

subject to:

$$\sum_{a \in i^- \cap \mathcal{A}} x_a^k + \sum_{a \in i^- \cap \mathcal{B}} x_a^k - \sum_{a \in i^+ \cap \mathcal{A}} x_a^k - \sum_{a \in i^+ \cap \mathcal{B}} x_a^k = \begin{cases} -1 & \text{if } i = o^k \\ 1 & \text{if } i = d^k \\ 0 & \text{otherwise} \end{cases} \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{N} \qquad (26)$$

$$\lambda_{h(a)}^k - \lambda_{t(a)}^k \leq c_a + t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (27)$$

$$\lambda_{h(a)}^k - \lambda_{t(a)}^k \leq c_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{B} \qquad (28)$$

$$\sum_{a \in \mathcal{A}} (c_a x_a^k + p_a^k) + \sum_{a \in \mathcal{B}} c_a x_a^k = \lambda_{d_k}^k - \lambda_{o_k}^k \qquad \forall k \in \mathcal{K} \qquad (29)$$

$$p_a^k \leq M_a^k x_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (30)$$

$$t_a - p_a^k \leq N_a (1 - x_a^k) \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (31)$$

$$p_a^k \leq t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (32)$$

$$p_a^k \geq 0 \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (33)$$

$$x_a^k \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (34)$$

$$x_a^k \geq 0 \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{B}, \qquad (35)$$

where $h(a)$, $t(a)$ correspond to the head and tail of toll arc $a \in \mathcal{A}$, while M_a^k and N_a are large constants. Constraints (26) describe flows on commodities. (27), (28) and (29) are the primal dual constraints and optimality conditions of the lower level problem. Constraints (30), (31) and (32) ensure that $p_a^k = t_a x_a^k$ for all $k \in \mathcal{K}, a \in \mathcal{A}$.

Alternatively, Heilporn et al. [21] show that the optimality of the lower level problem can be expressed in terms of path flows, without resorting to dual variables. Upon introduction of the sets $\mathcal{P}^k$ of paths associated with commodity $k \in \mathcal{K}$, the primal-dual optimality conditions (27), (28) and (29) of (NP2) can be replaced by

$$\sum_{a \in \mathcal{A}} (c_a x_a^k + p_a^k) + \sum_{a \in \mathcal{B}} c_a x_a^k \leq \sum_{a \in \mathcal{A} \cap p} (c_a + t_a) + \sum_{a \in \mathcal{B} \cap p} c_a \qquad \forall k \in \mathcal{K}, \forall p \in \mathcal{P}^k, \qquad (36)$$

which impose that the cost of an optimal path for a commodity $k \in \mathcal{K}$ is smaller than the cost of any

other path for the associated commodity.

The Network Pricing Problem has been investigated by several researchers. From the theoretical standpoint, Roch et al. [34] and Grigoriev et al. [17] have proved its $\mathcal{NP}$-hardness, under various restrictive conditions. However, some particular cases are polynomially solvable, such as the Network Pricing Problem with a single toll arc (see Brotcorne et al. [4]). Actually, Van Hoesel et al. [37] showed that, when the number of toll arcs is bounded, the optimal solution can be obtained by solving a polynomial number of linear programs. The latter authors also consider other polynomially solvable variants.

In contrast with the 'arc formulation' (NP2), Bouhtou et al. [2] and Didi Biha et al. [13] have proposed formulations involving path flow variables. These are based on a graph reduction whose size is in practice much smaller than that of the original graph, and allows for the exact solution of medium-size instances (up to 80 commodities, 100 toll arcs and 4000 toll-free arcs) within a couple of seconds. Note however that the randomly generated instances involved on average less than 2 to 3 paths per origin-destination pairs, thus making for combinatorially 'easy' problems.

Unfortunately, no off-the-shelf software can address the above formulations for large scale instances, mainly due to the poor quality of the upper bound obtained by relaxing the integrality requirements in the single-level formulations. To overcome this difficulty, several avenues have been investigated. By computing upper bounds on the toll arcs, Dewez et al. [12] obtain tight values for the constants $M_a^k, N_a : k \in \mathcal{K}, a \in \mathcal{A}$ in formulation (NP2), while simultaneously introducing valid inequalities for both the arc and path formulations. Numerical tests have been performed on randomly generated instances involving grid graphs, where high number of available paths, together with their interactions, increase the numerical challenge. The authors show that the proposed bounds allow to halve the duality gap at the root node of the branch-and-bound tree, whereas the valid cuts allow a further reduction of the number of explored nodes as well as the computing time.

Other improvements can be achieved by focusing on the efficient resolution of the inverse problem, which consists of finding revenue maximizing tolls compatible with a given flow assignment. Since the latter possesses the structure of a side-constrained flow problem, it is amenable to column generation (see Cirinei [7]). Tests on random grid networks indicate that the method significantly speeds up the solution process. Coupled with an efficient generation of the lower level solutions and a clever use of data structures, the algorithm also improves sharply the upper bounds on the revenue.

For larger instances, Brotcorne et al. [5] presents two heuristics: a quick and greedy method that sets tolls sequentially over the arcs, and a primal-dual approach based on penalizing the complementarity constraints that occur when the lower level problem is replaced by its primal-dual optimality conditions. Tested on random grids, these approaches yielded solutions that lied between 7% and 1% of

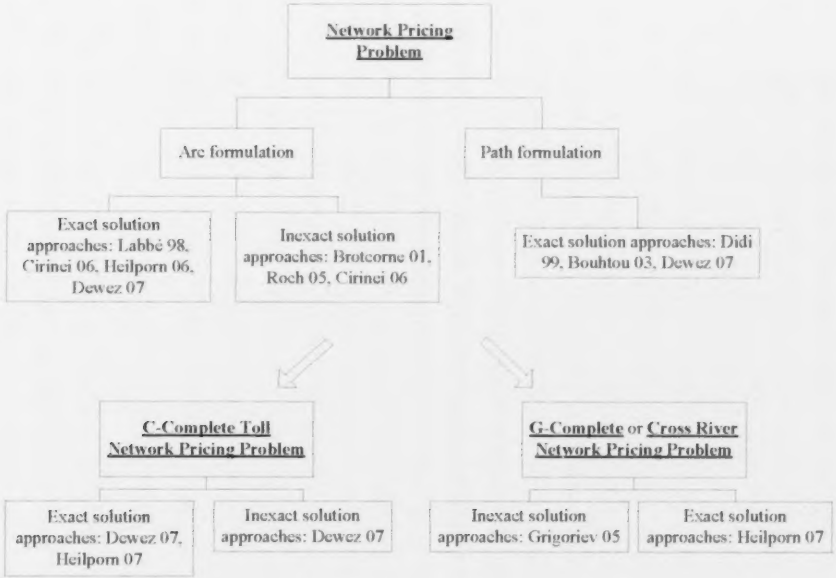
Network Pricing Problem
Arc formulation
Path formulation
Exact solution approaches: Labbé 98, Cirinei 06, Heilporn 06, Dewez 07
Inexact solution approaches: Brotcorne 01, Roch 05, Cirinei 06
Exact solution approaches: Didi 99, Bouhtou 03, Dewez 07
C-Complete Toll Network Pricing Problem
G-Complete or Cross River Network Pricing Problem
Exact solution approaches: Dewez 07, Heilporn 07
Inexact solution approaches: Dewez 07
Inexact solution approaches: Grigoriev 05
Exact solution approaches: Heilporn 07

Figure 4: Main contributions to the Network Pricing Problem

optimality, respectively. A similar approach was applied by Brotcorne et al. [4] in the framework of a single-commodity transportation problem. From a different perspective, Cirinei [7] implemented a tabu metaheuristic that exploits the network structure of the lower level problem, and could produce solution within 1% of optimality in reasonable computing time.

Let us also mention the approximation algorithm of Roch et al. [34] for the single-commodity Network Pricing Problem, that achieves a performance guarantee of $\frac{1}{2}\log n + 1$, where n is the number of toll arcs in the network.

Finally, Brotcorne et al. [6] address an extension of the Network Pricing Problem where the leader must simultaneously determine which toll arcs belong to the network. They propose for its solution a Lagrangian based heuristic, and obtain near-optimality solutions for random generated instances of medium sizes.

We close this literature review with a schematic view (see Figure 1) of contributions to the Network Pricing Problem.

4 Relationships between models

The common thread to welfare, profit and toll problems lies in the explicit consideration of rational, utility-maximizing customers. While, in most models presented in Section 2, the set of products is exogenous, this is not the case for the network pricing problem where products corresponds to paths, whose number is exponentially large. This has an impact both on the model formulation and on the numerical resolution. However, there is a clear parallel between the Seller Welfare Problem and the Highway Network Pricing Problem. Both problems involve revenue maximization in the face of either utility-maximizing purchasers or cost-minimizing travellers. Next, the Full Profit Problem, where all products are offered, is equivalent to the Highway Network Pricing Problem. Indeed, one can match purchaser segments with commodities, and products with toll arcs. The product prices $\pi_i : i \in \mathcal{I}$ then correspond to the tolls $t_a : a \in \mathcal{A}$, while the reservation price r_i^k of purchaser k for obtaining product i corresponds to the taxation window $c_{od}^k - c_a^k$. This is summarized in Table 1 below. Hence, while a purchaser segment buys the product that maximizes its utility $r_i^k - \pi_i$, a commodity travels on the toll arc that maximizes the difference $c_{od}^k - c_a^k - t_a$, i.e., that minimize its travel cost $c_a^k + t_a$.

Modified Profit Problem	**Highway Network Pricing Problem**
Purchaser segments $k \in \mathcal{K}$	Commodities $k \in \mathcal{K}$
Products $i \in \mathcal{I}$	Toll arcs $a \in \mathcal{A}$
Reservation prices $r_i^k : k \in \mathcal{K}, i \in \mathcal{I}$	Gains $c_{od}^k - c_a^k : k \in \mathcal{K}, a \in \mathcal{A}$
Prices $p_i : i \in \mathcal{I}$	Tolls $t_a : a \in \mathcal{A}$
Flows $x_i^k : k \in \mathcal{K}, i \in \mathcal{I}$	Flows $x_a^k : k \in \mathcal{K}, a \in \mathcal{A}$

Table 1: Links between notations for the Full Profit Problem and the Highway Network Pricing Problem

Similar relationships can be established between the Bundle Pricing Problem, where each product to be priced represents a subset of components of a 'global element', and the Network Pricing Problem, where each arc can be considered to be the component of a path (product). However, the connection is not as direct as for the previous two problems.

The two families of problems have inspired different algorithmic approaches. While exact resolution methods are often proposed for the Network Pricing Problem and its variants, those are almost nonexistent in the field of product pricing. In the network pricing family, several authors have proved valid inequalities or related polyhedral results for the problem, together with numerical tests proving the efficiency of such optimization tools. They also consider that the problem involves two hierarchical decision levels, yielding

a bilevel model that can be transformed in a single mixed integer program. In the Product Pricing family, the authors have focused on heuristic methods, with the only exceptions of McBryde and Zufryden [31] and Shioda et al. [36], although these authors do not go so far in the polyhedral structure of the problem.

Finally, from a theoretical viewpoint, the complexity of both families of problems has been investigated by several researchers. Table 2 provides an overview of the complexity results obtains for the various classes of problem.

Network Pricing Problem (NPP)	$\mathcal{NP}$-hard (Grigoriev et al. [17], Roch et al. [34])
with lower bounded tolls	$\mathcal{NP}$-hard (Labbé et al. [27])
with unrestricted tolls	$\mathcal{NP}$-hard (Roch et al. [34])
with a single commodity	$\mathcal{NP}$-hard (Roch et al. [34])
with a single toll arc	Polynomial (Brotcorne et al. [5])
with number of toll arcs upper bounded	Polynomial (van Hoesel et al. [37])
Cross River NPP	$\mathcal{NP}$-hard (Grigoriev et al. [17])
Highway NPP	$\mathcal{NP}$-hard (Heilporn et al. [20])
with a single commodity	Polynomial (Dewez [11])
Buyer's Welfare Problem	$\mathcal{NP}$-hard (Dobson and Kalish [15])
Profit Problem	$\mathcal{NP}$-hard (Dobson and Kalish [15])
Share-of-Choices Problem	$\mathcal{NP}$-hard (Kohli and Krishnamurti [23])
Bundle Pricing Problem	$\mathcal{APX}$-hard (Guruswani et al. [18])

5 Conclusion

In this paper we have highlighted the close relationship between two related problems that have been studied in economics, operations research and theoretical computer science, in the hope that breakthrough achieved in either field will lead to improved algorithms for addressing design and pricing problems in fields such as industrial economics or revenue management. For instance, Triangle and Monotonicity inequalities that occur naturally in highway pricing could be integrated in the general product pricing problem. Indeed, when prices are assigned to product with different product formats, it would make sense, whenever the product quantity X satisfies the relationship $X = Y + Z$, to require the triangle inequality $\pi_X \leq \pi_Y + \pi_Z$, for the sake of market consistency. In the same vein, one would expect that $\pi_X \leq \pi_Y$ if $X \leq Y$, i.e., Monotonicity inequalities are satisfied.

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